Pairings for Cryptography

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Pairings

A pairing is a bilinear, non-degenerate map

$$e : G_1 \times G_2 \to G_3,$$

where $(G_1, +), (G_2, +), (G_3, \cdot)$ are abelian groups.

- *bilinear*:

$$\begin{aligned} e(P_1 + P_2, Q_1) &= e(P_1, Q_1)e(P_2, Q_1), \\ e(P_1, Q_1 + Q_2) &= e(P_1, Q_1)e(P_1, Q_2), \end{aligned}$$

  i.e. $e(aP, Q) = e(P, Q)^a = e(P, aQ)$, $a \in \mathbb{Z}$.
- *non-degenerate*: given $0 \neq P \in G_1$ there is a $Q \in G_2$ with $e(P, Q) \neq 1$.

Cryptographic applications require e to be efficiently computable and the DLPs in G_1, G_2, G_3 to be hard.

Applications of pairings in cryptography

- Attack DL-based cryptography on elliptic curves (Menezes-Okamoto-Vanstone-1993, Frey-Rück-1994) .
- Construct crypto systems with certain special properties:
  - One-round tripartite key agreement (Joux-2000),
  - Identity-based, non-interactive key agreement (Ohgishi-Kasahara-2000),
  - Identity-based encryption (Boneh-Franklin-2001),
  - Hierarchical IBE (Gentry-Silverberg-2002),
  - Short signatures (Boneh-Lynn-Shacham-2001),
  - Searchable encryption (Boneh-Di Crescenzo-Ostrovsky-Persiano-2004),
  - Non-interactive proof systems (Groth-Sahai-2008),
  - much more ...

Elliptic curves

Take an elliptic curve E over $\mathbb{F}_q$ ($\mathrm{char}(\mathbb{F}_q) = p > 3$) with

- Weierstrass equation

$$E : y^2 = x^3 + ax + b,$$

- $E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q^2 : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$,
- $n = \#E(\mathbb{F}_q) = q + 1 - t, \quad |t| \leq 2\sqrt{q}$,
- and $r \mid n$ a large prime divisor of n ($r \neq p$).
- For $\mathbb{F} \supseteq \mathbb{F}_q$:
  $E(\mathbb{F}) = \{(x, y) \in \mathbb{F}^2 : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$,
- $E = E(\overline{\mathbb{F}_q})$, $\overline{\mathbb{F}_q}$ an algebraic closure of $\mathbb{F}_q$.
- E is an abelian group (written additively) with neutral element $\mathcal{O}$.

Torsion points and embedding degree

The set of r-torsion points on E is

$$E[r] = \{P \in E \mid [r]P = \mathcal{O}\} \cong \mathbb{Z}/r\mathbb{Z} \times Z/r\mathbb{Z}.$$

Since $r \mid \#E(\mathbb{F}_q)$, we have $E(\mathbb{F}_q)[r] \neq \emptyset$.
The embedding degree of E w.r.t. r is the smallest integer k with

$$r \mid q^k - 1.$$

For $k > 1$ we have

$$E[r] \subset E(\mathbb{F}_{q^k}),$$

i. e. $E(\mathbb{F}_q)[r] \subseteq E(\mathbb{F}_{q^k})[r] = E[r]$.

The reduced Tate pairing

Let $k > 1$. The reduced Tate pairing

$$\begin{aligned} t_r : E(\mathbb{F}_{q^k})[r] \times E(\mathbb{F}_{q^k})/[r]E(\mathbb{F}_{q^k}) &\longrightarrow \mu_r \subseteq \mathbb{F}_{q^k}^*, \\ (P, Q) &\longmapsto f_{r,P}(Q)^{\frac{q^k-1}{r}} \end{aligned}$$

is a non-degenerate, bilinear map, where

- $f_{r,P}$ is a function with divisor $(f_{r,P}) = r(P) - r(\mathcal{O})$,
- μ_r is the subgroup of r-th roots of unity in $\mathbb{F}_{q^k}^*$.

The computation of the pairing has two stages:

- evaluation of the Miller function $f_{r,P}$ at Q,
- the final exponentiation to the power $(q^k - 1)/r$.

Specific parameters for crypto

- k should be small,
- DLPs in all groups must be hard,
- for efficiency reasons balance the security.

Security level (bits)	Extension field size of q^k (bits)	EC base point order r (bits)	ratio $\rho \cdot k$
80	1248	160	7.8
112	2432	224	10.9
128	3248	256	12.7
192	7936	384	20.7
256	15424	512	30.1

ECRYPT II recommendations (2009), $\rho = \log(q)/\log(r)$.

Small embedding degree

The embedding degree condition says

$$r \mid q^k - 1,\ r \nmid q^m - 1,\ m < k$$

or

$$q^k \equiv 1 \pmod r,\ q^m \not\equiv 1 \pmod r,\ m < k.$$

This means:

- k is the (multiplicative) order of q modulo r,
- $k \mid r - 1$.

There are only $\varphi(k) < k$ elements of order k mod r. Given r and q, it is very unlikely that q is one of them.
(Note: r has at least 160 bits.)

Pairing-friendly curves

Fix a suitable value for k and find primes r, p and a number n with the following conditions:

- $n = p + 1 - t$, $|t| \leq 2\sqrt{p}$,
- $r \mid n$,
- $r \mid p^k - 1$,
- $t^2 - 4p = Dv^2 < 0$, $D, v \in \mathbb{Z}$, $D < 0$ squarefree, $|D|$ small enough to compute the Hilbert class polynomial in $\mathbb{Q}(\sqrt{D})$.

Given such parameters, a corresponding elliptic curve over $\mathbb{F}_p$ can be constructed by the CM method.

See Freeman, Scott, and Teske (*A taxonomy of pairing-friendly elliptic curves*) for an overview of construction methods and recommendations.

MNT curves and Freeman curves

- MNT curves (2001): $\rho \approx 1$ and $k \in \{3, 4, 6\}$.

k	$p(u)$	$t(u)$
3	$12u^2 - 1$	$-1 \pm 6u$
4	$u^2 + u + 1$	$-u$ or $u + 1$
6	$4u^2 + 1$	$1 \pm 2u$

- Freeman curves (2006): $\rho \approx 1$ and $k = 10$.

$$\begin{aligned} p(u) &= 25u^4 + 25u^3 + 25u^2 + 10u + 3, \\ t(u) &= 10u^2 + 5u + 3. \end{aligned}$$

- In both families, curves are very rare. Need to solve a Pell equation to find curves.
- D is variable.

BN curves

(Barreto-N., 2005)

If $u \in \mathbb{Z}$ such that

$$\begin{aligned} p = p(u) &= 36u^4 + 36u^3 + 24u^2 + 6u + 1, \\ n = n(u) &= 36u^4 + 36u^3 + 18u^2 + 6u + 1 \end{aligned}$$

are both prime, then there exists an ordinary elliptic curve

- with equation $E : y^2 = x^3 + b,\ b \in \mathbb{F}_p$,
- $r = n = \#E(\mathbb{F}_p)$ is prime, i. e. $\rho \approx 1$,
- the embedding degree is $k = 12$,
- $t^2 - 4p(u) = -3(6u^2 + 4u + 1)^2$.

BN curves are ideal for the 128-bit security level.

Specific parameters

Security level (bits)	Family	r (bits)	k	ρ	$\rho \cdot k$	p^k (bits)
80	MNT	208	6	1.00	6	1248
112	Fre	244	10	1.00	10	2440
128	BN	256	12	1.00	12	3072
192	KSS	384	16	1.25	20	7680
192	KSS	384	18	1.33	24	9216
256	Cyc	512	24	1.25	30	15360

Three groups

In practice, restrict the arguments of the Tate pairing to groups of prime order r.
Assume $r^2||\#E(\mathbb{F}_{p^k})$, $k > 1$. Define:

- $G_1 = E(\mathbb{F}_{p^k})[r] \cap \ker(\phi_p - [1]) = E(\mathbb{F}_p)[r]$,
- $G_2 = E(\mathbb{F}_{p^k})[r] \cap \ker(\phi_p - [p])$,
- $G_3 = \mu_r \subset \mathbb{F}_{p^k}^*$.

ϕ_p is the p-power Frobenius on E, i. e. $\phi_p(x, y) = (x^p, y^p)$. It is $E(\mathbb{F}_{p^k})[r] = G_1 \oplus G_2$.

- If $P \in E(\mathbb{F}_p)[r]$, then $t_r(P, P) = 1$. Take $Q \notin \langle P \rangle = G_1$.
- Can compute the Tate pairing on $G_1 \times G_2$ or on $G_2 \times G_1$.

Two choices

- The reduced Tate pairing:

$$\begin{aligned} t_r : G_1 \times G_2 &\rightarrow G_3, \\ (P, Q) &\mapsto f_{r,P}(Q)^{\frac{p^k-1}{r}}. \end{aligned}$$

- The ate pairing: Let $T = t - 1$.

$$\begin{aligned} a_T : G_2 \times G_1 &\rightarrow G_3, \\ (Q, P) &\mapsto f_{T,Q}(P)^{\frac{p^k-1}{r}}. \end{aligned}$$

Miller's algorithm (Tate)

Input: $P \in G_1, Q \in G_2, r = (r_m, \ldots, r_0)_2$

Output: $t_r(P,Q) = f_{r,P}(Q)^{\frac{p^k-1}{r}}$

$R \leftarrow P$, $f \leftarrow 1$

for $(i \leftarrow m-1;\ i \geq 0;\ i--)$ **do**

$\quad f \leftarrow f^2 \frac{l_{R,R}(Q)}{v_{[2]R}(Q)}$

$\quad R \leftarrow [2]R$

$\quad$ **if** $(r_i = 1)$ **then**

$\quad\quad f \leftarrow f \frac{l_{R,P}(Q)}{v_{R+P}(Q)}$

$\quad\quad R \leftarrow R + P$

$\quad$ **end if**

end for

$f \leftarrow f^{\frac{p^k-1}{r}}$

return f

Miller's algorithm (ate)

Input: $P \in G_1, Q \in G_2, T = (t_m, \ldots, t_0)_2$

Output: $a_T(P, Q) = f_{T,Q}(P)^{\frac{p^k-1}{r}}$

$R \leftarrow Q$, $f \leftarrow 1$

for $(i \leftarrow m-1;\ i \geq 0;\ i--)$ **do**

$f \leftarrow f^2 \frac{l_{R,R}(P)}{v_{[2]R}(P)}$

$R \leftarrow [2]R$

if $(t_i = 1)$ **then**

$f \leftarrow f \frac{l_{R,Q}(P)}{v_{R+Q}(P)}$

$R \leftarrow R + Q$

end if

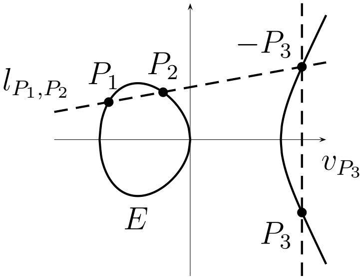
end for

$f \leftarrow f^{\frac{p^k-1}{r}}$

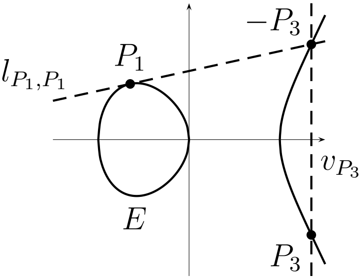
return f

Line functions

- Line functions correspond to the lines in the point doubling/addition,
- l_{P_1,P_2}: line through P_1 and P_2, tangent if $P_1 = P_2$, v_{P_3}: vertical line through $P_3 = P_1 + P_2$.



(a) addition



(b) doubling

The final exponentiation

Let Φ_d be the d-th cyclotomic polynomial.

- We have
$$X^k - 1 = \prod_{d|k} \Phi_d(X).$$
- $r \mid p^k - 1,\ r \nmid p^d - 1$ for $d < k \iff r \mid \Phi_k(p)$.
- Write the final exponent as:

$$\frac{p^k - 1}{r} = \prod_{d|k,\ d\neq k} \Phi_d(p) \cdot \frac{\Phi_k(p)}{r}.$$

Let $e \mid k,\ e \neq k$, then $\alpha^{(p^k-1)/r} = 1$ for all $\alpha \in \mathbb{F}_{p^e}$ since $(p^e - 1) \mid \prod_{d|k,\ d\neq k} \Phi_d(p)$.

Factors in proper subfields of $\mathbb{F}_{p^k}$ are mapped to 1 by the final exponentiation.

The final exponentiation (k even)

$$\frac{p^k - 1}{r} = (p^{k/2} - 1)\frac{p^{k/2} + 1}{\Phi_k(p)} \cdot \frac{\Phi_k(p)}{r}.$$

- Use $\mathbb{F}_{p^k} = \mathbb{F}_{p^{k/2}}(\alpha)$, $\alpha^2 = \beta$, β a non-square in $\mathbb{F}_{p^{k/2}}$.
  For $f = f_0 + f_1\alpha \in \mathbb{F}_{p^k}$: $(f_0 + f_1\alpha)^{p^{k/2}} = f_0 - f_1\alpha$,
  and $(f_0 + f_1\alpha)^{p^{k/2}-1} = (f_0 - f_1\alpha)/(f_0 + f_1\alpha)$.
- $(p^{k/2} + 1)/\Phi_k(p)$ is a sum of p-powers, use the p-power Frobenius automorphism.
  $k = 12 : f^{(p^6+1)/r} = f^{(p^2+1)\cdot\frac{p^4-p^2+1}{r}} = ((f^p)^p \cdot f)^{(p^4-p^2+1)/r}$.
- The last part is done with multi-exponentiation or by finding a good addition chain for $\Phi_k(p)/r$.

Using a twist to represent G_2

Here: A twist E' of E is a curve isomorphic to E over $\mathbb{F}_{p^k}$.

- A twist is given by
  $E' : y^2 = x^3 + (a/\omega^4)x + (b/\omega^6),\ \omega \in \mathbb{F}_{p^k}$
  with isomorphism

$$\psi : E' \to E,\ (x', y') \mapsto (\omega^2 x', \omega^3 y').$$

- If E' is defined over $\mathbb{F}_{p^{k/d}}$ and ψ is defined over $\mathbb{F}_{p^k}$ and no smaller field, d is called the degree of E'.
- Define $G_2' := E'(\mathbb{F}_{p^{k/d}})[r]$, then $\psi : G_2' \to G_2$ is a group isomorphism.
- Points in G_2 have a special form.

Maximal possible twist degrees

d	$j(E)$ a, b	fields of definition for powers of ω
2	$\notin \{0, 1728\}$ $a \neq 0$, $b \neq 0$	$\omega^2 \in \mathbb{F}_{q^{k/2}}$ $\omega^3 \in \mathbb{F}_{q^k} \setminus \mathbb{F}_{q^{k/2}}$
4	1728 $a \neq 0$, $b = 0$	$\omega^4 \in \mathbb{F}_{q^{k/4}}$, $\omega^2 \in \mathbb{F}_{q^{k/2}}$ $\omega^3 \in \mathbb{F}_{q^k} \setminus \mathbb{F}_{q^{k/2}}$
6	0 $a = 0$, $b \neq 0$	$\omega^6 \in \mathbb{F}_{q^{k/6}}$, $\omega^3 \in \mathbb{F}_{q^{k/3}}$ $\omega^2 \in \mathbb{F}_{q^{k/2}}$

$$E' : y^2 = x^3 + (a/\omega^4)x + (b/\omega^6)$$

$$\psi : E' \to E,\ (x', y') \mapsto (\omega^2 x', \omega^3 y')$$

Advantages of using twists

If E has a twist of degree d and $d \mid k$:

- Replace all curve arithmetic in G_2 (over $\mathbb{F}_{p^k}$) by curve arithmetic in G_2' (over $\mathbb{F}_{p^{k/d}}$)
- For $d > 2$, curve arithmetic is faster since $a = 0$ or $b = 0$.
- For even k, the x-coordinates of points in G_2 lie in $\mathbb{F}_{p^{k/2}}$, i. e. the vertical line function values $v_{P_3}(Q) = x_Q - x_3$ lie in $\mathbb{F}_{p^{k/2}}$ and can be omitted.
- Can use the twisted ate pairing ($e = k/d$ and $T_e = (t-1)^e \mod r$):

$$\eta_{T_e} : G_1 \times G_2 \to G_3,\ (P, Q) \mapsto f_{T_e,P}(Q)^{(p^k-1)/r}.$$

For $d > 2$, can have $\log(T_e) < \log(r)$.

Loop shortening

There are several possibilities to reduce the number of iterations in Miller's algorithm:

- Can take $T_e^j \mod r$ for $1 \leq j \leq d-1$ instead of T_e in the twisted ate pairing. Choose the shortest non-trivial power.
- For the ate pairing, can replace T by $T^j \mod r$ for $1 \leq j \leq k-1$ to possibly get a shorter loop.
- More combinations are possible, often leading to optimal pairings with a minimal loop length of $\log(r)/\varphi(k)$.
- For BN curves, the R-ate pairing is optimal:

$$R(Q,P) = \Big(f_{c,Q}(P)(f_{c,Q}(P)l_{[c]Q,Q}(P))^p \cdot l_{\phi_p([c]Q+Q),[c]Q}(P)\Big)^{(p^{12}-1)/n},$$

where $c = 6u + 2$.

Line functions for ate pairings

$$f \leftarrow f \cdot l_{R,Q}(P), \quad R \leftarrow R + Q$$

Do curve arithmetic in Miller's algorithm in G_2'. Replace points $R, Q \in G_2$ by corresponding points $R', Q' \in G_2'$.

- Using the slope on the twist:

$$\lambda = \frac{y_R - y_Q}{x_R - x_Q} = \frac{\omega^3 y_{R'} - \omega^3 y_{Q'}}{\omega^2 x_{R'} - \omega^2 x_{Q'}} = \omega \frac{y_{R'} - y_{Q'}}{x_{R'} - x_{Q'}} = \omega\lambda'$$

- Computing the line function on the twist:

$$\begin{aligned} l_{R,Q}(P) &= y_R - y_P - \lambda(x_R - x_P) \\ &= \omega^3 y_{R'} - \omega^3 y_{P'} - \omega\lambda'(\omega^2 x_{R'} - \omega^2 x_{P'}) \\ &= \omega^3 (y_{R'} - y_{P'} - \lambda(x_{R'} - x_{P'})) = \omega^3 \cdot l_{R',Q'}(P') \end{aligned}$$

Choice of coordinates

For “real” implementations, one tries to avoid inversions by using projective coordinates.

- Can do pairing computation with only 1 finite field inversion (needed in the final exponentiation).
- Can avoid inversions completely when using compressed representation of pairing values.
- The best choice of coordinates is different for different classes of curves.
- For the fastest explicit formulas to compute the DBL and ADD steps in Miller’s algorithm on curves with twists of degree $d > 2$, see preprint *Faster Pairing Computations on Curves with High-Degree Twists* (joint work with Craig Costello and Tanja Lange, will be out soon).

Thanks for your attention

- Database and web interface to get and compute parameters of BN curves:

  http://www.ti.rwth-aachen.de/research/cryptography/bncurves.php
- C-Implementation of several pairings on BN curves: http://www.cryptojedi.org/crypto

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